

Multi-fractal analysis of weighted networks

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Abstract

In many real complex networks, the fractal and self-similarity properties have been found. The fractal dimension is a useful method to describe fractal property of complex networks. Fractal analysis is inadequate if only taking one fractal dimension to study complex networks. In this case, multifractal analysis of complex networks are concerned. However, multifractal dimension of weighted networks are less involved. In this paper, multifractal dimension of weighted networks is proposed based on box-covering algorithm for fractal dimension of weighted networks (BCANw). The proposed method is applied to calculate the fractal dimensions of some real networks. Our numerical results indicate that the proposed method is efficient for analysis fractal property of weighted networks.

Keywords: multifractality, box-covering algorithm, weighted networks

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1. Introduction

Complex networks have been studied in various fields, including computer science, physics, management science, biology, etc[1, 2, 3, 4, 5, 6, 7]. Some fundamental properties of real complex networks have attracted much attention, such as small-world phenomena [8], scale-free degree [9] and community structure [10] etc. In 2005, fractal and self-similarity properties of complex networks have been investigated by Song et al [11]. The fractal dimension is used for fractal analysis of complex networks. The box-covering algorithm is described in detail and applied to calculate the fractal dimension of many real networks [12, 13]. Subsequently, the classical box-covering algorithm for complex networks is extensively used by many researchers [14, 15, 16, 17, 18, 19]. However, fractal analysis is inadequate when complex networks are studied by a single fractal dimension. In this case, multifractal analysis is a useful way to systematically characterize the spatial heterogeneity of both theoretical and experimental fractal patterns [20, 21, 22]. Multifractal has been applied successfully in many different fields such as financial modeling [23, 24], time series analysis [25] and complex networks [11, 26, 27]. For fractal analysis of complex networks, Wang et al.[20] introduced an improved box-covering algorithm for multifractal analysis of complex networks, a family of fractal networks is studied by Li et al. [28]. However, the existing works mainly focus on handling the fractal dimension of unweighted networks. An improved box-covering algorithm is used for fractal dimension of weighted networks (BCANw) in Ref [29]. By adopting the weighted box-covering algorithm, the

fractal property of weighted networks can be well described. In this paper, an algorithm of multifractal of weighted networks is proposed, and multifractal property of some real weighted networks is revealed.

In the following sections, BCANw and multifractal method are introduced in section 2. The proposed model of multifractal analysis for complex networks is described in section 3. In section 4, multifractal dimensions of some real complex networks are calculated by the proposed method. Some conclusions are presented in section 5.

2. Preliminaries

2.1. Box-covering algorithm for weighted networks

In this section, a box-covering algorithm for weighted networks (BCANw) is briefly introduced. Given a weighted network $G = (N, V, W)$, $N = (1, 2, \dots, n)$ is a set of nodes and $V = (1, 2, \dots, m)$ is a set of edges. $W = (1, 2, \dots, m)$ is a set of edge-weight and denoted by w_{ij} . w_{ij} is value of edge-weight. The network G is unweighted when the cell $x_{ij}(i, j = 1, 2, \dots, n)$ of edge is equal to 1 if node i is connected to node j , and 0 otherwise. G is defined as a weighted network if w_{ij} could be any real numbers. For unweighted networks, the shortest path between node i and node j is defined as follows.

Definition 2.1 (shortest path in unweighted networks). Denoting d_{ij} as the shortest path of between node i and node j , which satisfies

$$d_{ij} = \min(x_{ih} + \dots + x_{hj}) \quad (1)$$

There are two cases in weighted networks that needs notice. One case is that higher the weights, larger the distance of shortest path. The other is the opposite. Shortest path of the weighted networks is uniform defined as follows [29].

Definition 2.2 (the shortest path in the weighted networks). Denoting d_{ij} as the shortest path of between node i and node j , which satisfies any of the following conditions

$$d_{ij} = \min(w_{ij_1}^p + w_{j_1j_2}^p + \cdots + w_{j_{m-1}j_m}^p + w_{j_mj}^p) \quad (2)$$

where $j_m(m = 1, 2, \cdots)$ are IDs of nodes and p is a real number.

The weighted networks have different edge weights, which can be non-integers. Thus, values of d_{ij} may be non-integers too. It is also probable that the value of d_{ij}^{max} is less than one. values of the box size neither increase by one in turn nor the initial value equal 1. Fractal property of weighted networks cannot be reversed by the classical box-covering algorithm of complex networks [13]. An improved box-covering algorithm for weighted networks is obtained in Ref [29]. It is obtained by stacking the value of distance until the value is more than the value of $d_{ij}^{max}(i, j = 1, 2, \cdots, n)$. The detailed algorithm of BCANw is given as follows. Firstly, values of d_{ij} between node i and node j connected directly are obtained. Suppose the order of d_{ij} is denoted by $d_1 < d_2 < \cdots < d_m$. And then, value of l_B is a set and denoted by D , where $D = d_1, d_1 + d_2, \cdots, \sum_i^m d_i$. Finally, for a given box size l_B , node i and node j are connected if $d_{ij} \geq l_B$. In the following steps, it is similar to the classical box-covering algorithm of complex network. The proposed box-covering algorithm for the weighted networks is a modified version of coloring algorithm. Value of box size is obtained by stacking the value of distance. It can

directly calculate the fractal dimension of the weighted networks, and more corresponding points between box size and number are obtained.

2.2. Multifractal method

Multifractal analysis is a useful way to describe fractal property of complex system [20, 21, 22, 30]. For a set E in a metric space, a probability measure $0 \leq \mu \leq 1$ is given. A partition sum is considered as follows:

$$Z_\varepsilon(q) = \sum_{\mu(b)} [\mu(b)]^q \quad (3)$$

where q is a real number and $\mu(b)$ is $\mu(\cdot)$ of different non-overlapping boxes b which cover E with a give size ε . The mass exponent function of μ is denoted $\tau(q)$ and defined as follows:

$$\tau(q) = \lim_{\varepsilon \rightarrow 0} \frac{\ln Z_\varepsilon(q)}{\ln \varepsilon} \quad (4)$$

The generalize fractal dimension of the measure μ is denoted D_q which is defined as follows:

$$D_q = \frac{\tau(q)}{q-1}, q \neq 1 \quad (5)$$

Specially, $D_1 = \lim_{\varepsilon \rightarrow 0} \frac{Z_{1,\varepsilon}}{\ln \varepsilon}$. A view of life from the other side of multifractal, multifractal is defined as follows:

$$f(\alpha) = -\frac{\ln N(\alpha)}{\ln \varepsilon} \quad (6)$$

where $N(\alpha)$ is a number of box in $[\alpha, \alpha + d\alpha]$. It means that $f(\alpha)$ is fractal dimension of all boxes with α . There is Legendre transform between $f(\alpha)$ and $\tau(q)$ [31] as follows:

$$f(\alpha) = q\alpha(q) - \tau(q) \quad (7)$$

where $\alpha(q) = \frac{d\tau(q)}{dq}$

3. Multifractal analysis of weighted networks

3.1. Multifractal algorithm of weighted networks

Some algorithms of multifractal analysis are introduced [32, 28, 20]. For multifractal of complex networks, $\mu(b)$ is given as $\mu(b) = \frac{N_b}{N}$, where N_b is the number of nodes covered by the box b and N is the number of nodes. For a given box size r , the number of boxes is obtained by BCANw. And then, values of $\mu(b)$ is obtained. We calculate the partition sum as $Z_r(q) = \sum_{\mu(b) \neq 0} [\mu(b)]^q$ for each value of r . In order to obtain the generalized fractal dimension D_q , linear regression is an essential step for the appropriate range of box size. In this paper, the linear regression of $\ln Z_r(q)/(q-1)$ against $\ln(r/d)$ ($q \neq 1$) is considered. Last, values of D_q are obtained for various values of q . We determine the multifractality of weighted network by analyzing the shape of D_q .

3.2. Multifractal analysis of some real weighted networks

In this section, some real weighted networks - the USAir97 network, the C.elegans network and the Scientific collaboration network - are considered. The details of these weighted networks are obtained in Ref[29]. $\ln Z_r(q)/(q-1)$ against $\ln(r/d)$ ($q \neq 1$) of these networks are considered, shown in Figure 1.

In Figure 1, the value range of r is obtained by choosing appropriate points. Value of horizontal coordinates of USAir97 is $[-6, -2.5]$. In the

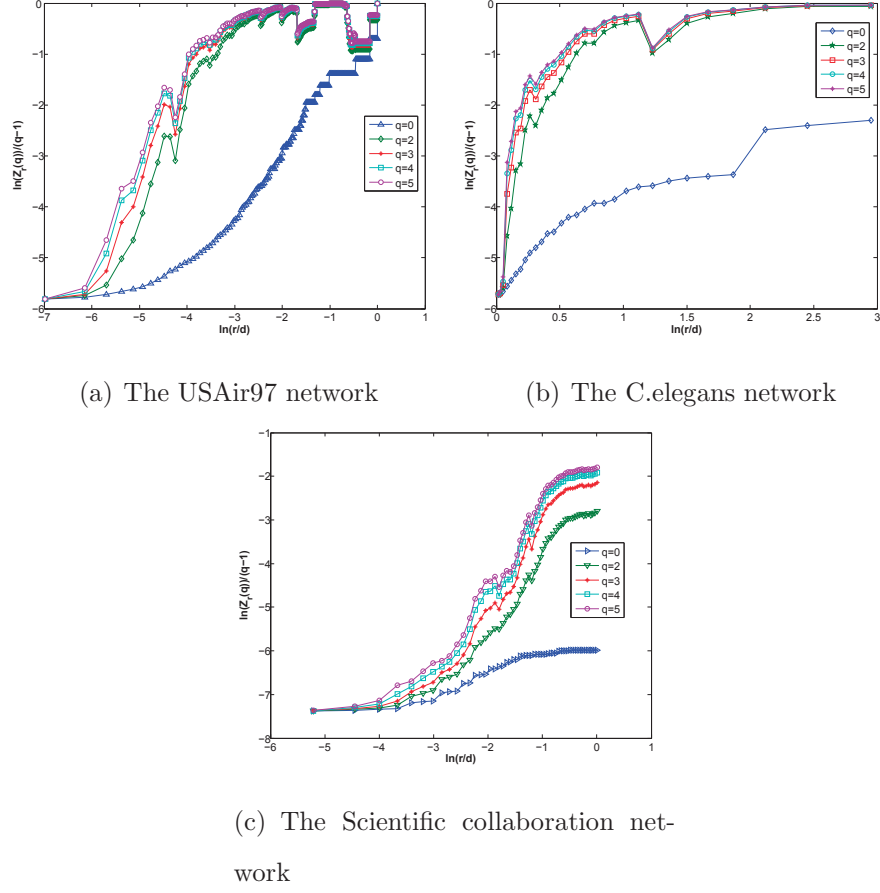


Figure 1: Multifractal scaling analysis of these real complex networks.

C.elegans network and the Scientific collaboration network, we adopt the points from 11th to 30th and from 5th to 20th, respectively. According to these, the relationship between D_q and q is considered and shown in Figure 2. In Figure 2, the values of D_q in all the three weighted networks increase when $q \in [0, 2]$, and reach their peak values around $q = 2$ for the USAir97 network and the C.elegans network. However, the maximum value of D_q of the Scientific collaboration network is the interval $[3, 4]$.

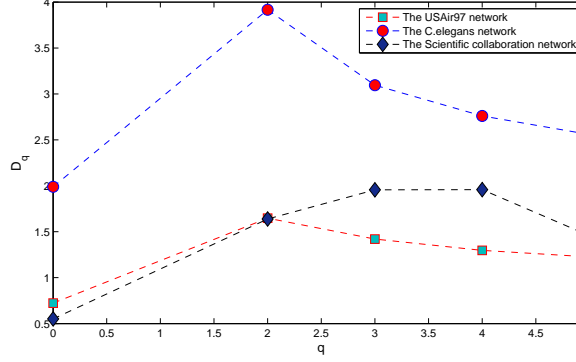


Figure 2: The D_q curves of the USAir97 network, the C.elegans network and the Scientific collaboration network.

4. Conclusions

Recently, research on complex networks have shown that some real network exhibit the property of fractal scaling. Some different physical quantities are considered in the definitions of fractal including the multifractal analysis of complex networks. However, the existing studies mainly deal with the fractal property of unweighted networks. In this paper, a multifractal analysis method for weighted networks has been proposed based on BCANw algorithm [29]. The numerical example of real weighted networks shows that the proposed approach can well reveal fractal property of weighted networks. To sum up, the proposed method is capable to reveal the fractal property of complex networks.

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